The Involutorial Birational Transformation of the Plane, of Order 17.

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§ 1. *Introduction.*

1. It has been shown by Bertini* that there are four distinct types of birational (Cremona) transformations in the plane that can not be transformed into simpler types, that are involutorial. From considerations of the Cremona net, the images of the lines of the plane, these are found to be:

(a) Harmonic homology.
(b) Jonquières's (perspective) transformation.
(c) Geiser's transformation.
(d) A transformation of order 17.

Of these four types the first three were known before, but type (d) was new. No equations are given, and but few of its properties are enumerated. In a foot-note a method of making the transformation geometrically is described † and attributed to Cremona. The contents of the note have been amplified by Sturm, ‡ and a summary of the above results are given by Doehlemann without further discussion. § This transformation is closely connected with that of the locus of the ninth double point of a net of sextics having eight fixed double points. ||

It is the purpose of this paper to derive the equations of the transformation of order 17, and to discuss some of its properties.

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* "Ricerche sulle trasformazioni univoche involutorie nel piano," *Ann. di Mat.*, Ser. II, Vol. VIII (1877), pp. 244–286.

† *l. c.*, p. 273.

‡ *Lehre von den geometrischen Verwandtschaften*, Vol. IV (1908), pp. 105–107 (No. 829).

§ *Geometrische Transformationen*, Vol. II (1908), pp. 165–174.

|| Halphen: "Sur les courbes planes du sixième degré à neuf points doubles," *Bulletin Soc. Math. de France*, Vol. X (1882), pp. 162–172.

§ 2. *Analytic Discussion.*

2. A rational curve of order 6 has 10 double points, but they can not be assigned arbitrarily. Indeed, if nine double points be chosen arbitrarily, the sextic through them will consist of the cubic determined by them counted twice, hence not more than eight double points can be chosen arbitrarily. Since these conditions are all linear when the positions of the double points are given, it follows that the sextics having eight given double points form a triply infinite linear system. The ninth double point must belong to some locus. A pencil of cubics pass through the eight given points, and they have one residual basis-point, which can not be a double point on the sextic, for a cubic of the pencil can pass through any tenth point. If this tenth point be chosen on the sextic, the cubic determined by it would have 19 points of intersection with the sextic, hence would be a factor of it.

3. Let f be a sextic having nine double points, and c a cubic passing through them. Every sextic of the pencil

$$c^2 + \lambda f = 0$$

will have these same nine points for double points. Hence: *If nine points are double points on a proper sextic curve, through them can be passed a pencil of sextics, each having the nine points for double points.* The pencil of sextics can not be transformed into a simpler one by any Cremona transformation.

4. Let the eight given points be denoted by $P_1, P_2, \ldots, P_8$, and ϕ, ψ denote two cubics passing through them, and f denote a sextic having them all for double points. The system

$$\alpha\phi^2 + \beta\phi\psi + \gamma\psi^2 + \delta f = 0 \tag{1}$$

represents a triply infinite linear system of sextics having P_i for double points. From the fact that any sextic of the set is completely fixed by three additional points, it follows that the above equation comprises every sextic having P_i for double points.

Let P' be any point in the plane. The curves of (1) which pass through P' form a net, the equation of which may be written

$$\alpha(\phi^2 f' - \phi'^2 f) + \beta(\psi^2 f' - \psi'^2 f) + \gamma(\phi\psi f' - \phi'\psi' f) = 0,$$

wherein ϕ' etc. denote the result of substituting the coordinates of P' in ϕ etc. By equating to zero the coefficients of $\alpha, \beta, \gamma,$ we obtain

$$\frac{\phi^2}{\phi'^2} = \frac{\psi^2}{\psi'^2} = \frac{\phi\psi}{\phi'\psi'} = \frac{f}{f'},$$

from which we see that all the curves of the net pass through all the intersections of

$$\left.\begin{aligned} \phi\psi' - \phi'\psi &= 0, \\ \phi\psi f' - \phi'\psi' f &= 0. \end{aligned}\right\} \quad (2)$$

The two curves of (2) intersect in 18 points, of which 16 are in P_i and one at P'. The residual P'' is fixed when P' is given. Hence:

In a triply infinite linear system of sextic curves having eight common fixed double points, the curves of the net determined by a further basis-point P' will all pass through another fixed point P''.

Since the coordinates of P' and of P'' enter the equations in exactly the same way, the birational transformation defined by them must be involutorial. The equations can be found from (2) by solving for the residual point of intersection. The image of a straight line l is obtained by eliminating the coordinates x_i from its equation and from (2). From the resultant of order 24, $\phi'\psi'$ divides out as a factor, and the linear expression involving P'. Hence:

By our involutorial transformation straight lines go into rational curves of order 17.

The equations of the transformation are believed to be new; the order 17 was determined by Bertini from the relations connecting fundamental curves of a Cremona net and by Cremona by means of the depiction of a cubic surface on a plane. From the method of elimination it follows immediately that P_i is a sixfold point on c_{17}, the image of an arbitrary line. The fundamental curves are therefore sextics. Since each must be rational and since any two can intersect only in fundamental points, each has a triple point at the fundamental point to which it corresponds, and double points at the remaining basis-points. These curves uniquely determine the transformation.

§ 3. *Geometric Interpretation.*

5. We may obtain the same result by considering the sextic having a triple point at P_1 and double points at each of the other basis-points, for which the symbol

$$f_6\,(1^3\,2^2\,3^2\,4^2\,5^2\,6^2\,7^2\,8^2)$$

will be used. If for P' we take any point on this curve, the curve of the pencil $\phi\psi f' - \phi'\psi' f = 0$ remains fixed while the associated cubic has one parameter. Thus any cubic intersects f_6 in 17 fixed points, and the image of any point P' on f_6 is therefore P_1 or 1. Since all points P_i enter symmetrically, the Jacobian of the system is of order 48; hence the order of the transformation is 17.

6. The ninth basis-point of the pencil of cubics is not a fundamental point of the transformation, nor are the two residual intersections of ϕ with f or of ψ with f, since ϕ', ψ' divide out of the equations of the transformation. This fact illustrates a general principle which can be applied in a number of similar cases. Consider the pencil of lines $l+\lambda m=0$ and of conics $c+\mu k=0$. Any point P' will uniquely determine a line $lm'-l'm=0$ and a conic $ck'-c'k=0$. The line and conic intersect in P' and in one other point P'' uniquely fixed by P'. The pairs of points P', P'' determine an involution and define a birational transformation. Evidently the lines passing through (l, m) and any basis-point of the pencil of conics are simple fundamental lines belonging to the fundamental point through which they pass, respectively. In the same way we may consider the fundamental curve belonging to the point (l, m). The result is the conic of the pencil determined by (l, m). The configuration of fundamental curves being of order 6, the transformation is the Jonquières transformation of order 3, and may be defined by a cubic curve and a point P upon it. Any line through P cuts the curve in two points A_1, A_2. The image of any point P' is the harmonic conjugate of P' as to A_1, A_2 on the line PP'. The fundamental curve of P is its polar conic as to the cubic, and the fundamental lines are the tangents to the cubic from P.

7. Now suppose one conic of the pencil to be composite, one factor passing through (l, m). This requires that two basis-points lie on a straight line containing (l, m). This line now divides out as a factor. The fundamental conic is replaced by a straight line and the transformation becomes quadratic inversion.

8. If the pencil of conics is of the form

$$c+\mu lm=0,$$

both l, m divide out as factors and no fundamental curves remain. The transformation is now harmonic homology with (l, m) as center and its polar as to c as axis.

The second case can be derived from the third by transforming through an inversion having one vertex at the center of the homology, and neither of the others on the axis.

9. Another illustration is furnished by the pencil of conics and a pencil of quartics, the latter having double points at three of the basis-points of the conics, but not passing through the fourth. The transformation is now of order 8, but reduces to 4 when the pencil of quartics is of the form $c_2 k_2+\mu c_4=0$, when the pencil of conics is $c_2+\lambda k_2=0$. The residual basis-point of the pencil of conics is not a fundamental point of the transformation.

10. Every involutorial birational transformation in the plane can be defined by two pencils of curves having a sufficient number of common basis-points to account for all but two intersections of a curve of one pencil and one of the other. By associating those curves of the two pencils which pass through a point P', the residual intersection P'' can be found by rational operations. Since every curve has a series g_2', it follows that such systems must always be hyperelliptic, elliptic or rational. From known theorems regarding the canonical form of hyperelliptic curves we again arrive at Bertini's theorem that all involutorial birational transformations can be transformed into one of the four types mentioned above. Interesting particular cases can be obtained by choosing the fundamental points of the transforming operator in special positions.

§ 4. *Curve of Invariant Points.*

11. Every Cremona transformation possesses a number of invariant points; when the transformation is involutorial these form a locus. This locus of invariant points must be distinguished from an invariant curve which has its points exchanged in pairs by the transformation. The former points will appear whenever $P' \equiv P''$. In our case this will occur when a cubic touches the associated sextic. Since a net of sextics pass through P_i, P', P'', it follows that at least one curve of the system has a double point at P', and is therefore a factor of the Jacobian of the net. After a few reductions this is found to be

$$f'(\phi\psi' - \phi'\psi)^2 \frac{\partial(\phi, \psi, f)}{\partial(x_1, x_2, x_3)} = 0.$$

Since f is a fixed curve and P' can be chosen at will, the factors f, $\phi\psi' - \phi'\psi$ may be rejected; hence the proper locus is

$$J \equiv \frac{\partial(\phi, \psi, f)}{\partial(x_1, x_2, x_3)} = 0.$$

Any point of J can be a ninth double point on a proper sextic of the system except the critical centers of the pencil $\phi + \lambda\psi = 0$. These points are double points on curves of the pencil of cubics; for them $P' \equiv P''$ and $J = 0$; but if k is a critical center the sextic having this point as a double point will meet the associated cubic in 20 points, that is, the sextic must degenerate.

The coordinates of these points satisfy the relations

$$\frac{\phi_1}{\psi_1} = \frac{\phi_2}{\psi_2} = \frac{\phi_3}{\psi_3},$$

wherein ϕ_i denotes differentiation of ϕ as to x_i.

12. The Jacobian J is of order 9; it evidently passes through P_j since f has a double point at each basis-point. It will now be shown that J has a triple point at each basis-point of the system. If we differentiate J as to x_i, we obtain:

$$J_i \equiv \begin{vmatrix} \phi_{1i} & \psi_1 & f_1 \\ \phi_{2i} & \psi_2 & f_2 \\ \phi_{3i} & \psi_3 & f_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & \psi_{1i} & f_1 \\ \phi_2 & \psi_{2i} & f_2 \\ \phi_3 & \psi_{3i} & f_3 \end{vmatrix} + \begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ \phi_3 & \psi_3 & f_{3i} \end{vmatrix}.$$

The first two determinants vanish at P_j since $f_i = 0$. In the third determinant, multiply the first row by x_1, the second by x_2, and the third by x_3, and add the products for a new third row. The result is:

$$x_3 J_i = \begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ 3\phi & 3\psi & 5f_i \end{vmatrix} = 0,$$

which shows that P_j is a double point on J.

Now compute the second derivatives as to x_i, x_k and retain only those determinants which do not vanish for the reasons mentioned above. We may write:

$$J_{ik} = (\phi_{1i}\,\psi_1\,f_{1k}) + (\phi_1\,\psi_{1i}\,f_{1k}) + (\phi_1\,\psi_1\,f_{1ik}) + (\phi_{1k}\,\psi_1\,f_{1k}) + (\phi_1\,\psi_{1k}\,f_{1i}).$$

By performing the same operations as before, we have:

$$x_3 J_{ik} = 2\phi_i \begin{vmatrix} \psi_1 & f_{1k} \\ \psi_2 & f_{2k} \end{vmatrix} - 2\psi_i \begin{vmatrix} \phi_1 & f_{1k} \\ \phi_2 & f_{2k} \end{vmatrix} + 4f_{ik} \begin{vmatrix} \phi_1 & \psi_1 \\ \phi_2 & \psi_2 \end{vmatrix} + 2\phi_k \begin{vmatrix} \psi_1 & f_{1i} \\ \psi_2 & f_{2i} \end{vmatrix} - 2\psi_k \begin{vmatrix} \phi_1 & f_{1i} \\ \phi_2 & f_{2i} \end{vmatrix},$$

which may be written in the form:

$$2\begin{vmatrix} \phi_1 & \psi_1 & f_{1k} \\ \phi_2 & \psi_2 & f_{2k} \\ \phi_i & \psi_i & f_{ik} \end{vmatrix} + 2\begin{vmatrix} \phi_1 & \psi_1 & f_{1i} \\ \phi_2 & \psi_2 & f_{2i} \\ \phi_i & \psi_i & f_{ii} \end{vmatrix}.$$

When $i = 1$ or $i = 2$, we have two identical rows; when $i = 3$, we can make the elements of the third row zero by performing the same operations as before. We now have the theorem:*

The locus of coincident points in the involutorial transformation of order 17 *is a curve of order* 9, *having a triple point at each basis-point. It is the locus of the ninth double point of a sextic curve having double points at eight given points.*

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* E. C. Valentiner: "Nogle Sätninger on visse algebraiske Kurven," *Zeuthen's Zeitschrift*, Ser. IV, Vol. V (1881), pp. 88–91.

13. A curve of order 9, having triple points at eight given points belongs to a linear system having six degrees of freedom. It may therefore be defined by an equation of the form

$$(\phi, \psi)_3 + (\phi, \psi)_1 f + kJ = 0,$$

wherein $(\phi, \psi)_i$ is a binary form of order i, and k is a constant. In general, the curves of this system which pass through a point P' will define a system of five degrees of freedom and having no further point in common. If, however, the P' be chosen so that the curves defined by the equations

$$\phi\psi' - \phi'\psi = 0, \qquad \phi^2 f' - \phi'^2 f = 0, \qquad \phi^3 J' - \phi'^3 J = 0$$

meet in a point, then all the curves of the system of order 9 will also pass through another point P''. The locus of P' is a curve of order 18, having sixfold points at the basis-points P_j. The point P'' lies on the same curve, and the system of curves defines an involutorial transformation which leaves the curve as a whole invariant. Among the curves of order 9 having threefold points at P_j are those having a ninth triple point. This can not be chosen arbitrarily, for the conditions are all linear and one such curve is the cubic determined by the nine points, taken three times. After a ninth point has been found which is a triple point of a proper curve of the system, then every curve of the pencil

$$\phi^3 + \lambda c_9 = 0$$

will have nine triple points. This pencil can not be reduced to a simpler one by birational transformations. In this way a general elliptic system of curves of order $3r$ having nine r-fold points can be found. Of these, eight points may be chosen arbitrarily, and the ninth lies on a definite locus. If this point lies on J, every curve of the system remains invariant under our transformation T_{17}.

14. An arbitrary straight line c_1 goes into c_{17} by T_{17}. Since c_1 cuts J_9 in nine points through which the image c_{17} must pass, c_1 cuts c_{17} in eight other points, which must be arranged in pairs, since the transformation is involutorial. Caporali* has suggested the word class to define the number of pairs of points on an arbitrary straight line. *The involutorial transformation of order* 17 *is of class* 4. Apart from the basis-points, c_{17} and J_9 have only these nine points in common. Conversely, given any c_{17} having sixfold points at all the basis-points P_j. It will intersect J_9 in nine points which always lie on a straight line, the image of c_{17} in T_{17}.

* "Sulle trasformazioni univoche piane involutorie," *Memorie di Geometria*, Naples, 1888, pp. 116–125.

§ 5. *Transformation through Quadratic Inversions.*

15. If T_{17} be transformed through a quadratic inversion having all its vertices at basis-points, it will go into one of similar form, but if one of the vertices be an arbitrary point P_9, a new form will result. Let I_{129} denote an inversion having vertices at P_1, P_2, P_9. It will be convenient to indicate the basis-points by their subscripts. Let $\bar{9}$ be the image of 9 under T_{17} and ξ be the image of $\bar{9}$ under I_{129}. By using the notation $c_1 I_{129} c_2 (129)$ to indicate that the image of a straight line in I_{129} is a conic through the vertices 1, 2, 9, we may write

$$c_2(129)\, T_{17}\, c_{22}(1^7\, 2^7\, 3^8\, 4^8\, 5^8\, 6^8\, 7^8\, 8^8\, \bar{9}^1)\, I_{129}\, c_{30}(1^{15}\, 2^{15}\, 3^8\, 4^8\, 5^8\, 6^8\, 7^8\, 8^8\, 9^8\, \xi^1)$$

and

$$I_{129}\, T_{17}\, I_{129} = T_{30}.$$

The configuration of fundamental curves becomes:

$$\begin{array}{ll}
1: & c_{15}(1^8\, 2^7\, 3^4\, 4^4\, 5^4\, 6^4\, 7^4\, 8^4\, 9^4\, \xi^1), \\
2: & c_{15}(1^7\, 2^8\, 3^4\, 4^4\, 5^4\, 6^4\, 7^4\, 8^4\, 9^4\, \xi^1), \\
3: & c_8\ (1^4\, 2^4\, 3^3\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2\, 9^2), \\
\multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots,} \\
9: & c_8\ (1^4\, 2^4\, 3^2\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2\, 9^3), \\
\xi: & c_1\ (1\ 2).
\end{array}$$

The invariant curve J_9 becomes $J_{12}(1^6\, 2^6\, 3^3 \ldots . 9^3)$. If, however, the point 9 be chosen on J_9, the new transformation is of order 29, obtained from the preceding case by removing the factor $c_1(12)$ from c_{30}, each c_{15} and c_1, as well as suppressing ξ in the preceding form. J is now of order 11.

The class of T_{30} is 9, and so is T_{29}. *The class of an involution is not invariant under birational transformation.*

16. Geometrically, T_{30} can be obtained as follows: Let ϕ_4, ψ_4 be two quartic curves having double points at 1, 2, and passing simply through $3, \ldots, 9$. Let f_8 be an octic having fourfold points at 1, 2 and double points at $3, \ldots, 9$. Every quartic of the pencil $\phi_4 + \lambda \psi_4 = 0$ will have the same property as those described, as will also every octic of the pencil $\phi_4 \psi_4 + \mu f_8 = 0$. A quartic and an octic will intersect in two variable points, either of which uniquely determines the other. In addition to the basis-points, the images of the lines of the plane have one other point in common, in order to have two degrees of freedom and have only one variable point of intersection. The whole configuration can

be obtained from the preceding one by transforming the two systems of curves through the quadratic inversion. The necessary and sufficient condition that T_{29} results is that all nine basis-points can be double points on a proper sextic curve.

§ 6. *Particular Cases.*

17. If three of the basis-points, say 1, 2, 3, lie on a straight line, it must be a factor of c_{17} and of the fundamental curves of 1, 2, 3. The configuration becomes:

$$c_1\, T_{16} \,.\, c_{16}\,(1^5\, 2^5\, 3^5\, 4^6\, 5^6\, 6^6\, 7^6\, 8^6),$$
$$1: \quad c_5\,(1^2\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$
$$2: \quad c_5\,(1^1\, 2^2\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$
$$3: \quad c_5\,(1^1\, 2^1\, 3^2\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2),$$

and the others as in the general case. The proper curve of invariant points is now:

$$J_8\,(1^2\, 2^2\, 3^2\, 4^3\, 5^3\, 6^3\, 7^3\, 8^3).$$

These same results can be derived directly from the equations. Let c_1 denote the line joining 1, 2, 3 and c_2 the conic containing the other five basis-points. For ϕ_3 we may now write $c_1 c_2$ and for f_6 we can take $c_1 c_5$, wherein c_5 is defined by the symbol $c_5\,(1^1\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2)$.

The pencils analogous to (2) are now:

$$\psi_3' c_1 c_2 - c_1' c_2' \,.\, \psi_3 = 0,$$
$$c_2' \psi_3' \,.\, c_5 - c_5' \,.\, c_2 \psi_3 = 0.$$

To determine the fundamental curves, consider $c_5\,(1^2\, 2^1\, 3^1\, 4^2\, 5^2\, 6^2\, 7^2\, 8^2)$. If P' be chosen anywhere on this curve, P'' is evidently at 1, hence c_5 is the fundamental curve corresponding to 1. Similarly for the points 2 and 3. To determine the fundamental curve of 4, associate curves of the two pencils which touch each other at 4. The locus of the residual intersection is the required fundamental curve. It will have a fourfold point at 4, a double point at 1, 2 and 3 and triple points at 5, 6, 7, 8; it is of order 8. The locus must also pass through the residual intersection of $c_2\,(4, 5, 6, 7, 8)$ and ψ_3. As the former has 17 points on c_8, it must be a factor; the other component is $c_6\,(1^2\, 2^2\, 3^2\, 4^3\, 5^2\, 6^2\, 7^2\, 8^2)$, as in the general case. By performing the operations indicated in the definition of J, $c_1\,(1\;2\;3)$ appears as a factor, leaving J_8 as defined above.

18. If 1, 2, 3 lie on c_1 and 4, 5, 6 on k_1, then both lines appear twice in c_{17}. The transformation is now of order 13 and has the symbol

$$c_1\, T_{13}\, c_{13}\,(1^4\, 2^4\, 3^4\, 4^4\, 5^4\, 6^4\, 7^6\, 8^6).$$

The fundamental curves become

$$1: \quad c_4\,(1^2\,2^1\,3^1\,4^1\,5^1\,6^1\,7^2\,8^2),$$

and similarly for 2, ..., 6; those for 7, 8 are as in the general case. Since $\phi_3 = c_1 k_1 l_1$, $f_6 = c_1 k_1 c_4$, the equations of the generating pencils may be written in the form

$$\psi_3' \cdot c_1 k_1 l_1 - c_1' k_1' l_1' \cdot \psi_3 = 0,$$
$$l_1' \psi_3' \cdot c_4 - c_4' \cdot l_1 \psi_3 = 0.$$

The locus of coincident points has the symbol

$$J_7\,(1^2\,2^2\,3^2\,4^2\,5^2\,6^2\,7^3\,8^3).$$

The transformation is now of class 3.

19. In case two of the basis-points approach coincidence, forming a tacnode, the transformation can be reduced to that having three collinear basis-points by transforming through a quadratic inversion whose vertices are the tacnode and any other two basis-points. Conversely, from the three collinear points we can obtain the tacnode. Thus, if 1, 2, 3 are collinear, we may write

$$c_1\,I_{124}\,c_2\,(1\,2\,4)\;T_{16}\,c_{16}\,(1^5\,2^5\,3^6\,4^5\,5^6\,6^6\,7^6\,8^6)\;I_{124}\,c_{17}\,(1^6\,2^6\,\overline{3\,4}^6\,5^6\,6^6\,7^6\,8^6),$$

in which 3, 4 form a tacnode.

If in the same case we transform through I_{456}, we have

$$c_1\,I_{456}\,c_2\,(4\,5\,6)\;T_{16}\,c_{14}\,(1^4\,2^4\,3^4\,4^5\,5^5\,6^5\,7^6\,8^6)\;I_{456}\,c_{13}\,(1^4\,2^4\,3^4\,4^4\,5^4\,6^4\,7^6\,8^6),$$

the general Geiser transformation results.

in which the points 1, ..., 6 lie on a conic. A particular case was met with before when 1, 2, 3 lie on c_1 and 4, 5, 6 on k_1. The configuration of fundamental curves is the same as given for the two lines.

If finally T_{13} be transformed through I_{678}, thus:

$$c_1\,I_{678}\,c_2\,(6\,7\,8)\;T_{13}\cdot c_{10}\,(1^3\,2^3\,3^3\,4^3\,5^3\,6^2\,7^7\,8^8)\;I_{678}\,c_8\,(1^3\,2^3\,3^3\,4^3\,5^3\,7^3\,8^3),$$

the general Geiser transformation results.

20. Thus the three cases of a tacnode, three collinear points, and six points on a conic can all be transformed into each other by proper quadratic inversions, and each is equivalent to the general Geiser transformation (*c*). Further particularizations of the basis-points in T_{17} will therefore reduce to particular cases of the Geiser T_8. As the latter have already been treated at length* they need not be considered here.

CORNELL UNIVERSITY, *August*, 1910.

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* Snyder: "Conjugate Line Congruences Contained in a Bundle of Quadric Surfaces," *Transactions Amer. Math. Soc.*, Vol. XI (1910), pp. 371–387.